The Form of Even Perfect Numbers — An Independent Derivation

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Overview: The ancient Greek mathematicians had proven that if $\left(2^n - 1\right)$ is a prime number, then $2^{n-1} \times \left(2^n - 1\right)$ is a perfect number. About two thousand years later, Euler proved the converse of this theorem, i.e., every even perfect number must be of the above form.

However, I am not aware of the method that Euler used to prove it. Here I present my own independent method of proving that every even perfect number must be of the form given above.

Introduction: Let me first state what perfect numbers are. A perfect number is the sum of its factors greater than or equal to 1 but less than itself. E .g. — 6 (6 = 1 + 2 + 3), 28 (28 = 1 + 2 + 4 + 7 + 14).

Note that

$$
\begin{aligned}
6 &= 1 + 2 + 3 \\
&= \left(1 + 2^1\right) + 3 \times 1 \\
&= \left(2^2 - 1\right) + \left(2^2 - 1\right) \times 1 \quad \left[\because 1 + 2^1 = 3 = 2^2 - 1\right] \\
&= 2 \times \left(2^2 - 1\right) \\
&= 2^{2-1}\left(2^2 - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
28 &= 1 + 2 + 4 + 7 + 14 \\
&= \left(1 + 2^1 + 2^2\right) + 7\left(1 + 2^1\right) \\
&= \left(2^3 - 1\right) + \left(2^3 - 1\right)\left(2^2 - 1\right) \quad \left[\because 1 + 2^1 + 2^2 = 7 = 2^3 - 1\right] \\
&= \left(2^3 - 1\right)\left(1 + 2^2 - 1\right) \\
&= 2^2\left(2^3 - 1\right) \\
&= 2^{3-1}\left(2^3 - 1\right)
\end{aligned}
$$

Observations like these had led the ancient Greek mathematicians to prove that if $\left(2^n - 1\right)$ is a prime number, then $2^{n-1} \times \left(2^n - 1\right)$ is a perfect number. Observe that in the above examples, $\left(2^2 - 1\right) = 3$ and $\left(2^3 - 1\right) = 7$, which are prime numbers. Later, Euler proved the converse of this theorem, i.e., every even perfect number must be of the above form.

However, no one has ever discovered an odd perfect number. No one even knows whether an odd perfect number exists or not.

What I present here is my own independent method of proving that every even perfect number must be of the form

$$2^{n-1} \times \left(2^n - 1\right),$$

where $\left(2^n - 1\right)$ is prime.

The Proof:

An even number can have any one of the following forms:—

(a) 2^n, where n is a natural number

(b) $2^n \prod_{i=1}^{m} p_i^{l_i}$, where m and n are natural numbers,

p_i is a prime number other than 2, $\forall i = 1(1)m$,

l_i is a natural number, $\forall i = 1(1)m$,

$p_i \neq p_j$ for $i \neq j$,

l_i & l_j ($i \neq j$) may be same or different.

No even number of the form (a) can be a perfect number. $[\because 1 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 \neq 2^n]$

Now, let us consider the even numbers of the form (b) which satisfy the condition that if $m = 1$, then $l_1 \neq 1$.

Let, $\prod_{i=1}^{m} p_i^{l_i} = P$ .... (1) [if $m = 1$, then $l_1 \neq 1$]

If a number of the form (b) (which satisfies the condition that if $m = 1$, then $l_1 \neq 1$) has to be perfect, it must satisfy the equation

$2^n P = (1 + 2^1 + 2^2 + \cdots + 2^n) + (1 + 2^1 + 2^2 + \cdots + 2^n) \times$ (sum of all the factors of P that are greater than 1 but less than itself)

$+ P(1 + 2^1 + 2^2 + \cdots + 2^{n-1})$..................................... (2)

or, $2^n P = (1 + 2^1 + 2^2 + \cdots + 2^n) \times$ [1 + sum of all the factors of P that are greater than 1 but less than itself] + $P(1 + 2^1 + 2^2 + \cdots + 2^{n-1})$

$= (2^{n+1} - 1) \times$ (sum of all the factors of P that are greater than or equal to 1 but less than itself) + $P(2^n - 1)$

$= (2^{n+1} - 1)S + P(2^n - 1)$,

where S = sum of all the factors of P that are greater than or equal to 1 but less than itself ....(3)

or, $2^n P = 2^n P + \lfloor (2^{n+1} - 1)S - P \rfloor$ ....(4)

Equation (4) holds true if and only if

$$(2^{n+1} - 1)S - P = 0$$

$$\text{or, } P = (2^{n+1} - 1)S$$

$$\text{or, } \frac{P}{S} + 1 = 2^{n+1}$$

$$\text{or, } Q + 1 = 2^{n+1} \text{ ....(5),}$$

where $Q = \dfrac{P}{S}$ ....(6)

We note that equation (5) cannot hold true if Q is not an integer.

Let us investigate whether Q is an integer or not. From equation (6), it is evident that for Q to be an integer, S must be equal to any one of the factors of P. But from

equation (3), it is evident that S cannot be equal to any one of the factors of P that are greater than or equal to 1 but less than P.
Hence, there is only one possibility for which Q can be an integer and that is

$$S = P \text{ ....(7)}$$

Let us note that if for some P, equation (7) holds true, then that P is an odd perfect number.
However, we shall not go to investigate whether equation (7) is possible or not, since we do not need to do so for the current problem.
If at all equation (7) holds true for some value of P, then using equations (5), (6) & (7), we have

$$1 + 1 = 2^{n+1}$$

$$\text{or, } 2^1 = 2^{n+1},$$

which is impossible for $n \geq 1$.
Thus, we see that whatever be the value of P, equation (5) does not hold true for $n \geq 1$.
Hence, an even integer of the form (b) (which satisfies the condition that if $m = 1$, then $l_1 \neq 1$), cannot be perfect.
Now, we are left only with one form of even numbers and that is

$$2^n p \text{ ....(c),}$$

where n is a natural number
and p is a prime number other than 2.
Let us now investigate whether a number of the form (c) can be perfect or not.
If such a number has to be perfect, it must satisfy the equation

$$2^n p = \left(1 + 2^1 + 2^2 + \cdots + 2^n\right) + p\left(1 + 2^1 + 2^2 + \cdots + 2^{n-1}\right) \text{ ....(8)}$$

$$\text{or, } 2^n p = \left(2^{n+1} - 1\right) + p\left(2^n - 1\right)$$

$$\text{or, } 2^n p = 2^n p + \left[\left(2^{n+1} - 1\right) - p\right] \text{............(9)}$$

Equation (9) holds true if and only if

$$\left(2^{n+1} - 1\right) - p = 0$$

$$\text{or, } p = 2^{n+1} - 1 \text{..............(10)}$$

We note that certain numbers of the form $\left(2^{n+1} - 1\right)$ corresponding to certain values of n are prime. E.g. — $2^{1+1} - 1 = 3$ ($n = 1$), $2^{2+1} - 1 = 7$ ($n = 2$). It is also obvious that a number of this form can never be equal to 2 for $n \geq 1$. Hence, equation (10) is perfectly legitimate.
Thus, an even number of the form (c) is perfect if and only if the prime number p is of the form $\left(2^{n+1} - 1\right)$.
We have already shown that even numbers that are not of the form (c) cannot be perfect.
Hence, every even perfect number must be of the form

$$2^n\left(2^{n+1} - 1\right),$$

where n is a natural number
and $\left(2^{n+1} - 1\right)$ is prime.

Written in another form,

every even perfect number must be of the form

$$2^{n-1}\left(2^n - 1\right),$$

where n is a natural number greater than 1

and $\left(2^n - 1\right)$ is prime.

This completes the proof.

Reference:

1. *God Created The Integers: The Mathematical Breakthroughs That Changed History*, edited, with commentary, by Stephen Hawking, Penguin Books, page-5, 2006.